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2-BLOCKS AND 2-MODULAR CHARACTERS OF THE CHEVALLEY GROUPS $G_2(q)$

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ABSTRACT. We first determine the distribution of the ordinary irreducible characters of the exceptional Chevalley group $G_2(q)$, q odd, into 2-blocks. This is done by using the method of central characters. Then all but two of the irreducible 2-modular characters are determined. The results are given in the form of decomposition matrices. The methods here involve concepts from modular representation theory and symbolic computations with the computer algebra system MAPLE. As a corollary, the smallest degree of a faithful representation of $G_2(q)$, q odd, over a field of characteristic 2 is obtained.

1. INTRODUCTION

This paper continues the investigations of the modular characters of the finite Chevalley groups $G = G_2(q)$. In a series of papers [12, 13, 14, 15, 7, 8] the authors have investigated the p-blocks, the Brauer trees, and the p-modular characters for odd p. Here, we finally consider the case $p = 2$ and q odd.

We determine the distribution of the ordinary characters of G into 2-blocks and all but two of the irreducible Brauer characters. A complete solution seems to be beyond the scope of the methods of this paper. However, the results are sufficient to find the minimal degree of a faithful 2-modular representation of G. In [16], White has obtained similar results for the groups $\mathrm{Sp}_4(q)$.

Throughout the paper, we have to distinguish between the two cases $q \equiv 1 \pmod 4$ and $q \equiv -1 \pmod 4$. The ordinary characters of G are taken from [4, 3]. Our notation is that of Chang and Ree in [3]. The blocks are determined by using the method of central characters. With the help of lemmas from [13, 14] the distribution into blocks and the exceptional characters are calculated.

The methods for finding the decomposition matrices are much the same as those used in [7]. We determine a basic set of Brauer characters, consisting of some—but not all—of the nonexceptional characters in the block. Then we produce a large set of projective characters. The next step consists in finding a maximal linearly independent subset of these which approximates the projective indecomposables as closely as possible. This gives us a basic set of projectives. The projectives are now written in terms of this basic set. Those which require negative coefficients are used to refine the basic set.

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Unfortunately, one does not know a priori which characters to induce, or which characters to tensor with, to obtain the required information. So one has to produce as many projective characters as possible to begin with. This involves a huge amount of calculations, which would not be possible without computer support. In our case these calculations were done with the help of the computer algebra system MAPLE, which was developed by the Symbolic Computation Group at the University of Waterloo. Once MAPLE has done its job and the proofs are given in the form below (§4), they can in principle be checked by hand. So nobody has to worry about how these programs work, or about the internal data structure used to represent the characters. We are sure that other exceptional groups of Lie type can be dealt with in a similar way.

2. Results

2.1. **Explanation of the tables.** In this section we present the 2-modular decomposition numbers of $G = G_2(q)$, q odd. We note that the 2-blocks differ substantially from the p-blocks where $p \neq 2$ (see [8, 13, 14]). In particular, note the exceptional families in B_1. We start with some explanation of the tables. The ordinary characters in a block fall naturally into two distinct sets. The first set consists of the so-called nonexceptional characters. Its members all belong to a fixed geometric conjugacy class of characters (see [2, §12.1]). In case of the principal block, the nonexceptional characters are exactly the unipotent characters lying in the block. Their restrictions to the 2-regular conjugacy classes generate the ring of generalized Brauer characters, but in general are linearly dependent as class functions. However, a basic set can be selected from these. The decomposition of these basic set characters is given in the upper half of the decomposition matrix.

The remaining characters in a block are the so-called exceptional characters. They fall into families of characters which have the same restriction to 2-regular classes. Only one row is printed for any one family of exceptional characters in the lower half of the decomposition matrix. A family is indicated by curly brackets. The number of exceptionals in each family is printed in the last column of the decomposition matrix. The description of the exceptionals appears in §3.

The first column of the decomposition matrix gives the degrees of the ordinary characters. From these, the degrees of the irreducible Brauer characters in the block are easily derived. They are printed below the decomposition matrix.

In all of the following tables, missing entries are 0.

2.2. **The case** $4 \mid q-1$. Let $q \equiv 1 \pmod 4$. Denote $q-1 = 2^d \cdot r$ and $q+1 = 2r'$, r and r' odd. So, $d \geq 2$ and $|G_2(q)|_2 = 2d+2$.

2.2.1. *The principal block* B_1. Tables (a) and (b) give the decomposition matrix of the principal 2-block of G and the degrees of the irreducible Brauer characters, respectively.

Remarks. (i) $0 \leq \alpha \leq q-1$; if $3 \mid q$, we only get the weaker bound $0 \leq \alpha \leq 2q$.

(ii) $0 \leq \beta \leq \frac{1}{3}(q+2)$.

(iii) If 3 does not divide q, we have $\varphi_{12}(1) \geq \frac{1}{3}(q-1)^2(q+1)(q^3+2q^2+q+3)$.

(iv) If $3 \mid q$, we have $\varphi_{12}(1) \geq \frac{1}{3}(q-1)^2(q^3+2q^2+4q+3)$.

TABLE (a)

Degrees	Char.	φ_{11}	φ_{17}	φ_{18}	φ_{13}	φ_{14}	φ_{15}	φ_{12}	No. of Char.'s
1	X_{11}	1							1
$\frac{1}{2}q(q-1)^2(q^2+q+1)$	X_{17}		1						1
$\frac{1}{6}q(q-1)^2(q^2-q+1)$	X_{18}			1					1
$\frac{1}{3}q(q^4+q^2+1)$	X_{13}	1			1				1
$\frac{1}{3}q(q^4+q^2+1)$	X_{14}	1				1			1
$\frac{1}{2}q(q+1)^2(q^2-q+1)$	X_{15}		1				1		1
q^6	X_{12}	1	α	β	1	1		1	1
$\frac{1}{6}q(q+1)^2(q^2+q+1)$	X_{16}			1			1		1
(q^4+q^2+1)	$\{X_{22}\}$	1					1		1
$q(q^4+q^2+1)$	$\{X_{23}\}$	1	1	1		1	1		1
$q(q^4+q^2+1)$	$\{X_{24}\}$	1	1	1	1		1		1
$q^2(q^4+q^2+1)$	$\{X_{21}\}$	1	α	β	1	1	1	1	1
$(q+1)(q^4+q^2+1)$	$\{X'_{1a}\}$	2	1	1	1		2		$\frac{1}{2}(2^d-2)$
$q(q+1)(q^4+q^2+1)$	$\{X_{1a}\}$	2	$\alpha+1$	$\beta+1$	1	2	2	1	$\frac{1}{2}(2^d-2)$
$(q+1)(q^4+q^2+1)$	$\{X'_{1b}\}$	2	1	1		1	2		$\frac{1}{2}(2^d-2)$
$q(q+1)(q^4+q^2+1)$	$\{X_{1b}\}$	2	$\alpha+1$	$\beta+1$	2	1	2	1	$\frac{1}{2}(2^d-2)$
$(q^2-1)(q^4+q^2+1)$	$\{X_a\}$		α	β		2		1	$\frac{1}{4}2^d$
$(q^2-1)(q^4+q^2+1)$	$\{X_b\}$		α	β	2			1	$\frac{1}{4}2^d$
$(q+1)^2(q^4+q^2+1)$	$\{X_1\}$	4	$\alpha+2$	$\beta+2$	2	2	4	1	$\frac{1}{12}(2^d-4)(2^d-2)$

TABLE (b)

Char.	Degree
φ_{11}	1
φ_{17}	$\frac{1}{2}q(q-1)^2(q^2+q+1)$
φ_{18}	$\frac{1}{6}q(q-1)^2(q^2-q+1)$
φ_{13}	$\frac{1}{3}(q-1)(q^4+q^3+2q^2+2q+3)$
φ_{14}	$\frac{1}{3}(q-1)(q^4+q^3+2q^2+2q+3)$
φ_{15}	$q^2(q^2+1)$
φ_{12}	$\frac{1}{6}(q-1)^2(6q^4+(8-3\alpha-\beta)q^3+(10-3\alpha+\beta)q^2+(8-3\alpha-\beta)q+6)$

2.2.2. *The block B_3 in case $q \equiv 1 \pmod 3$*. For this case, Tables (c) and (d) give the decomposition matrix of B_3 and the degrees of the irreducible Brauer characters, respectively.

TABLE (c)

Degrees	Char.	φ_{32}	φ_{33}	φ_{31}	No. of Char.'s
q^3+1	X_{32}	1			1
$q(q+1)(q^3+1)$	X_{33}		1		1
$q^3(q^3+1)$	X_{31}	1		1	1
$q(q+1)(q^4+q^2+1)$	$\{X_{1a}\}$	1	1	1	2^d-1
$(q+1)(q^4+q^2+1)$	$\{X'_{1a}\}$	1	1		2^d-1
$(q^2-1)(q^4+q^2+1)$	$\{X_a\}$			1	$\frac{1}{2}2^d$
$(q+1)^2(q^4+q^2+1)$	$\{X_1\}$	2	2	1	$\frac{1}{6}(2^d-2)(2^d-1)$

TABLE (d)

Char.	Degree
φ_{32}	$(q+1)(q^2-q+1)$
φ_{33}	$q(q+1)^2(q^2-q+1)$
φ_{31}	$(q^2-1)(q^4+q^2+1)$

2.2.3. *The block B_3 in case $q \equiv -1 \pmod 3$*. For this case, Tables (e) and (f) give the decomposition matrix of B_3 and the degrees of the irreducible Brauer characters, respectively.

TABLE (e)

Degrees	Char.	φ_{32}	φ_{33}	φ_{31}	No. of Char.'s
q^3-1	X_{32}	1			1
$q(q-1)(q^3-1)$	X_{33}		1		1
$q^3(q^3-1)$	X_{31}	1	1	1	1
$q(q-1)(q^4+q^2+1)$	$\{X_{2b}\}$	1		1	1
$(q-1)(q^4+q^2+1)$	$\{X'_{2b}\}$	1	1		1
$(q^2-1)(q^4+q^2+1)$	$\{X_b\}$	2	1	1	2^d-1

TABLE (f)

Char.	Degree
φ_{32}	$(q-1)(q^2+q+1)$
φ_{33}	$q(q-1)^2(q^2+q+1)$
φ_{31}	$(q-1)^2(q^2+1)(q^2+q+1)$

Remark. The defect group of this block is a Sylow 2-subgroup of $SU_3(q)$, the special unitary group in three dimensions, and therefore is semidihedral of order 2^{d+2}. Blocks with such defect group and decomposition matrix have been considered in [6, Lemma 11.4].

2.2.4. *The blocks* B_{1a}. Tables (g) and (h) give the decomposition matrix for the blocks B_{1a} and the degrees of the irreducible Brauer characters, respectively.

TABLE (g)

Degrees	Char.	φ'_{1a}	φ_{1a}	No. of Char.'s
$(q+1)(q^4+q^2+1)$	X'_{1a}	1		1
$q(q+1)(q^4+q^2+1)$	X_{1a}	1	1	1
$(q+1)(q^4+q^2+1)$	$\{X'_{1a}\}$	1		2^d-1
$q(q+1)(q^4+q^2+1)$	$\{X_{1a}\}$	1	1	2^d-1
$(q^2-1)(q^4+q^2+1)$	$\{X_a\}$		1	$\frac{1}{2}2^d$
$(q+1)^2(q^4+q^2+1)$	$\{X_1\}$	2	1	$\frac{1}{2}2^d(2^d-1)$

TABLE (h)

Char.	Degree
φ'_{1a}	$(q+1)(q^4+q^2+1)$
φ_{1a}	$(q^2-1)(q^4+q^2+1)$

Number of blocks B_{1a}:
if $q \equiv 1 \pmod 3$: $\frac{1}{2}(r-3)$;
if $q \not\equiv 1 \pmod 3$: $\frac{1}{2}(r-1)$.

2.2.5. *The blocks* B_{1b}. Replace a by b in Tables (g) and (h).
Number of blocks B_{1b}: $\frac{1}{2}(r-1)$.

2.2.6. **The blocks** B_{2a}. These blocks have the decomposition matrix given in Table (i).

TABLE (i)

Degrees	Char.	φ'_{2a}	φ_{2a}	No. of Char.'s
$(q-1)(q^4+q^2+1)$	X'_{2a}	1		1
$q(q-1)(q^4+q^2+1)$	X_{2a}	1	1	1
$(q-1)(q^4+q^2+1)$	$\{X'_{2a}\}$	1		1
$q(q-1)(q^4+q^2+1)$	$\{X_{2a}\}$	1	1	1
$(q-1)^2(q^4+q^2+1)$	$\{X_2\}$		1	1
$(q^2-1)(q^4+q^2+1)$	$\{X_a\}$	2	1	2^d-1

The degrees of the irreducible Brauer characters are given in Table (j).
Number of blocks B_{2a}: $\frac{1}{2}(r'-1)$.

TABLE (j)

Char.	Degree
φ'_{2a}	$(q-1)(q^4+q^2+1)$
φ_{2a}	$(q-1)^2(q^4+q^2+1)$

Remark. The defect group of this block is a Sylow 2-subgroup of $U_2(q)$, the unitary group in two dimensions, and therefore is semidihedral of order 2^{d+2}. Blocks with such defect group and decomposition matrix have been considered in [5, Lemma 8.8].

2.2.7. **The blocks B_{2b}.** Replace a by b in Tables (i) and (j).
Number of blocks B_{2b}:
if $q \equiv -1 \pmod 3$: $\frac{1}{2}(r'-3)$;
if $q \not\equiv -1 \pmod 3$: $\frac{1}{2}(r'-1)$.

2.2.8. *The blocks B_{X_1}, B_{X_2}, B_{X_a}, and B_{X_b}.* B_{X_1}: Contains 2^{2d} characters of type X_1.
Number of blocks:
if $q \equiv 1 \pmod 3$: $\frac{1}{12}(r-3)^2$;
if $q \not\equiv 1 \pmod 3$: $\frac{1}{12}(r-1)(r-5)$.
B_{X_2}: Contains four characters of type X_2.
Number of blocks:
if $q \equiv -1 \pmod 3$: $\frac{1}{12}(r'-3)^2$;
if $q \not\equiv -1 \pmod 3$: $\frac{1}{12}(r'-1)(r'-5)$.
B_{X_α}, $\alpha = a$ or b: Contains 2^{d+1} characters of type X_α.
Number of blocks: $\frac{1}{8}(r-1)(q-1)$.
These are blocks with exactly one irreducible Brauer character, and so the decomposition matrix just consists of a column of 1's.

2.2.9. The characters X_{19}, $\overline{X}_{19}$ and the characters of types X_3, X_6 constitute blocks of defect 0.

2.3. **The case $4 \mid q+1$.** Let $q \equiv -1 \pmod 4$. Now denote $q+1 = 2^d \cdot r$ and $q-1 = 2r'$, r and r' odd. So, $d \geq 2$ and $|G_2(q)|_2 = 2d+2$.

2.3.1. *The principal block B_1.* The principal 2-block of G has the decomposition matrix given in Table (k).

Remarks. (i) $1 \leq \alpha \leq q-1$; if $3 \mid q$, we only get the weaker bound $1 \leq \alpha \leq 2q$.
(ii) $1 \leq \beta \leq \frac{1}{3}(q+2)$.
(iii) In case $q = 3$, we have $\beta = 1$. Ryba has shown, using some sophisticated extensions of Parker's MEAT-AXE, that $\alpha = 2$ in this case.
(iv) If $q > 3$, the degrees of the irreducible Brauer characters are the same as in the case $q \equiv 1 \pmod 4$.

TABLE (k)

Degrees	Char.	φ_{11}	φ_{17}	φ_{18}	φ_{13}	φ_{14}	φ_{15}	φ_{12}	No. of Char.'s
1	X_{11}	1							1
$\frac{1}{2}q(q-1)^2(q^2+q+1)$	X_{17}		1						1
$\frac{1}{6}q(q-1)^2(q^2-q+1)$	X_{18}			1					1
$\frac{1}{3}q(q^4+q^2+1)$	X_{13}	1			1				1
$\frac{1}{3}q(q^4+q^2+1)$	X_{14}	1				1			1
$\frac{1}{2}q(q+1)^2(q^2-q+1)$	X_{15}		1				1		1
q^6	X_{12}	1	α	β	1	1		1	1
$\frac{1}{6}q(q+1)^2(q^2+q+1)$	X_{16}			1			1		1
(q^4+q^2+1)	$\{X_{22}\}$	1					1		1
$q(q^4+q^2+1)$	$\{X_{23}\}$	1	1	1		1	1		1
$q(q^4+q^2+1)$	$\{X_{24}\}$	1	1	1	1		1		1
$q^2(q^4+q^2+1)$	$\{X_{21}\}$	1	α	β	1	1	1	1	1
$(q-1)(q^4+q^2+1)$	$\{X'_{2a}\}$		1	1		1			$\frac{1}{2}(2^d-2)$
$q(q-1)(q^4+q^2+1)$	$\{X_{2a}\}$		$\alpha-1$	$\beta-1$		1		1	$\frac{1}{2}(2^d-2)$
$(q-1)(q^4+q^2+1)$	$\{X'_{2b}\}$		1	1	1				$\frac{1}{2}(2^d-2)$
$q(q-1)(q^4+q^2+1)$	$\{X_{2b}\}$		$\alpha-1$	$\beta-1$	1			1	$\frac{1}{2}(2^d-2)$
$(q^2-1)(q^4+q^2+1)$	$\{X_a\}$		α	β		2		1	$\frac{1}{4}2^d$
$(q^2-1)(q^4+q^2+1)$	$\{X_b\}$		α	β	2			1	$\frac{1}{4}2^d$
$(q-1)^2(q^4+q^2+1)$	$\{X_2\}$		$\alpha-2$	$\beta-2$				1	$\frac{1}{12}(2^d-4)(2^d-2)$

2.3.2. *The block B_3 in case $q \equiv 1 \pmod 3$.* In this case, B_3 has the decomposition matrix given in Table (l).

The degrees of the irreducible Brauer characters are as in case 2.2.2.

TABLE (l)

Degrees	Char.	φ_{32}	φ_{33}	φ_{31}	No. of Char.'s
q^3+1	X_{32}	1			1
$q(q+1)(q^3+1)$	X_{33}		1		1
$q^3(q^3+1)$	X_{31}	1		1	1
$q(q+1)(q^4+q^2+1)$	$\{X_{1a}\}$	1	1	1	1
$(q+1)(q^4+q^2+1)$	$\{X'_{1a}\}$	1	1		1
$(q^2-1)(q^4+q^2+1)$	$\{X_a\}$			1	2^d-1

Remark. The defect group of this block is a Sylow 2-subgroup of $SL_3(q)$, the special linear group in three dimensions, and therefore is semidihedral of order 2^{d+2}. Blocks with such defect group and decomposition matrix have been considered in [6, Lemma 11.6].

2.3.3. *The block B_3 in case $q \equiv -1 \pmod 3$.* In this case, B_3 has the decomposition matrix given in Table (m).

TABLE (m)

Degrees	Char.	φ_{32}	φ_{33}	φ_{31}	No. of Char.'s
q^3-1	X_{32}	1			1
$q(q-1)(q^3-1)$	X_{33}		1		1
$q^3(q^3-1)$	X_{31}	1	γ	1	1
$q(q-1)(q^4+q^2+1)$	$\{X_{2b}\}$	1	$\gamma-1$	1	2^d-1
$(q-1)(q^4+q^2+1)$	$\{X'_{2b}\}$	1	1		2^d-1
$(q^2-1)(q^4+q^2+1)$	$\{X_b\}$	2	γ	1	$\frac{1}{2}2^d$
$(q-1)^2(q^4+q^2+1)$	$\{X_2\}$		$\gamma-2$	1	$\frac{1}{6}(2^d-2)(2^d-1)$

Remark. $1 \le \gamma \le \frac{1}{3}(q+1)$.

The degrees of the irreducible Brauer characters are given in Table (n).

TABLE (n)

Char.	Degree
φ_{32}	$(q-1)(q^2+q+1)$
φ_{33}	$q(q-1)^2(q^2+q+1)$
φ_{31}	$(q-1)^2(q^2+q+1)(q^2+(1-\gamma)q+1)$

2.3.4. **The blocks B_{1a}.** These blocks have the decomposition matrix given in Table (o).

TABLE (o)

Degrees	Char.	φ'_{1a}	φ_{1a}	No. of Char.'s
$(q+1)(q^4+q^2+1)$	X'_{1a}	1		1
$q(q+1)(q^4+q^2+1)$	X_{1a}	1	1	1
$(q+1)(q^4+q^2+1)$	$\{X'_{1a}\}$	1		1
$q(q+1)(q^4+q^2+1)$	$\{X_{1a}\}$	1	1	1
$(q+1)^2(q^4+q^2+1)$	$\{X_1\}$	2	1	1
$(q^2-1)(q^4+q^2+1)$	$\{X_a\}$		1	2^d-1

The degrees of the irreducible Brauer characters are as in case 2.2.4.

Remark. The defect group of this block is a Sylow 2-subgroup of $GL_2(q)$, the general linear group in two dimensions, and therefore is semidihedral of order 2^{d+2}. Blocks with such defect group and decomposition matrix have been considered in [5, Lemma 8.6].

2.3.5. *The blocks* B_{1b}. Replace a by b in Table (o).

2.3.6. *The blocks* B_{2a}. These blocks have the decomposition matrix given in Table (p).

TABLE (p)

Degrees	Char.	φ'_{2a}	φ_{2a}	No. of Char.'s
$(q-1)(q^4+q^2+1)$	X'_{2a}	1		1
$q(q-1)(q^4+q^2+1)$	X_{2a}	1	1	1
$(q-1)(q^4+q^2+1)$	$\{X'_{2a}\}$	1		2^d-1
$q(q-1)(q^4+q^2+1)$	$\{X_{2a}\}$	1	1	2^d-1
$(q^2-1)(q^4+q^2+1)$	$\{X_a\}$	2	1	$\frac{1}{2}2^d$
$(q-1)^2(q^4+q^2+1)$	$\{X_2\}$		1	$\frac{1}{2}2^d(2^d-1)$

The degrees of the irreducible Brauer characters are as in case 2.2.6.

2.3.7. *The blocks* B_{2b}. Replace a by b in Table (p).

2.3.8. *The blocks* B_{X_1}, B_{X_2}, B_{X_a}, *and* B_{X_b}. These are blocks with exactly one irreducible Brauer character, and so the decomposition matrix consists just of a column of 1's.

Corollary. *If* 3 *does not divide* q, *the smallest degree of a faithful representation of* G *over a field of characteristic* 2 *is* $q^3+\varepsilon$, *where* $\varepsilon=\pm 1$ *is such that* $q\equiv\varepsilon$ (mod 3). *If* $q=3^f$, $f\geq 2$, *the smallest degree is* $q^2(q+1)$, *and if* $q=3$, *the smallest degree is* 14.

3. PROOFS: BLOCKS

3.1. **Preliminaries.** As in [8, 13, 14], the blocks are determined by examination of the central character tables (mod 2). These are given in the Appendix for all characters of $G_2(q)$ of nonzero defect.

We rely on the fact that two characters are in the same 2-block if and only if they determine the same central character (mod 2).

3.1.1. We first recall facts and notation from [3, 14]:

$$|G_2(q)| = q^6(q^6-1)(q^2-1).$$

The following subgroups are the maximal tori of $G_2(q)$:

$$H_1 \cong C_{q-1}\times C_{q-1}, \qquad H_3 \cong C_{q^2+q+1},$$
$$H_2 \cong C_{q+1}\times C_{q+1}, \qquad H_6 \cong C_{q^2-q+1},$$
$$H_a \cong C_{q^2-1} \cong H_b.$$

For $\alpha\in\{1, 2, a, b, 3, 6\}$, denote elements of H_α by h_α, and complex linear characters of H_α by π_α; $\hat{\pi}_\alpha(h_\alpha)$ is the sum of the images under π_α of the conjugates of h_α in H_α. As in the previous papers, π_α will usually denote a character such that $X_\alpha(\pi_\alpha)$ is irreducible, π_α^+ are of order 3, and $\pi_\alpha^\times$ are of order 2, and I_α will denote the trivial character on H_α. For $\alpha = a$ or b, let the characters π_α^* and $\pi_\alpha^\#$ be such that $(\pi_\alpha^*)^{q+1} = I_\alpha = (\pi_\alpha^\#)^{q-1}$, and π_α^*, $\pi_\alpha^\#$ are of order > 3.

We say

$$\pi_1 \sim (i, j)$$

if $\pi_1(h_{1a}) = \rho^i$, $\pi_1(h_{1b}) = \rho^j$, where ρ is a primitive complex $(q-1)$st root of unity and $h_{1a} \in H_1 \cap H_a$, $h_{1b} \in H_1 \cap H_b$, and $|h_{1a}| = q-1 = |h_{1b}|$. Similarly for π_2, replace $q-1$ by $q+1$ and h_{1a}, h_{1b} by h_{2a}, h_{2b}.

3.1.2. We note that for primes $p \neq 2$, the central character tables $\pmod p$ contain nonzero entries for the unipotent class sums. This made it easier to determine the blocks. When two central characters coincided on the unipotent class sums, we had only to find criteria as to when $\hat{\pi}_\alpha(h) \equiv \hat{\pi}_\alpha(h) \pmod p$ for h in a maximal torus H_α.

For $p = 2$, however, we see that for all unipotent u we have $\omega_\chi(\hat{u}) \equiv 0 \pmod 2$ for all χ of nonzero defect. Hence, we need also to determine when

$$\hat{\pi}_\alpha(h) \equiv \hat{\pi}_\beta(h) \pmod 2$$

for $h \in H_\alpha \cap H_\beta$, H_α and H_β *different* tori.

To do this, we first find some π'_α such that $\hat{\pi}_\beta(h) \equiv \hat{\pi}'_\alpha(h) \pmod 2$, and then we use the lemmas in [14] to get all π_α such that $\hat{\pi}_\alpha(h) \equiv \hat{\pi}'_\alpha \pmod 2$.

The relevant lemmas (3.4, 5.3, and 6.3) in [14] still hold for $p = 2$. These essentially say that if χ is a character defined by π_α (for $\alpha \in \{a, b, 1, 2\}$), and we want to find all χ' defined by π'_α such that if $\langle h_\alpha \rangle = H_\alpha$ then

$$(*) \qquad \hat{\pi}_\alpha(h_\alpha^{p^d k}) \equiv \hat{\pi}'_\alpha(h_\alpha^{p^d k}) \pmod p \quad \text{for all } k,$$

then it is sufficient to find all χ' defined by π'_α such that

$$\pi_\alpha(h) \equiv \pi'_\alpha(h) \pmod p \quad \text{for all } p\text{-regular } h \text{ in } H_\alpha.$$

We note that an error appears in the proof of Lemma 3.4 in [14] which needs correction to allow the proof to work for $p = 2$.

We needed to show that for $\alpha = a$ or b, the congruence $(*)$ holds if and only if $P_\sigma(x) \equiv P_\tau(x) \pmod p$. Here, $P_\sigma(x) = (x-\sigma)(x-\sigma^{-1})(x-\sigma^q)(x-\sigma^{-q})$, $\sigma = \pi_\alpha(h_\alpha^{p^d})$, and $\tau = \pi'_\alpha(h_\alpha^{p^d})$. Then, if $A(\sigma) = \sigma + \sigma^{-1} + \sigma^q + \sigma^{-q}$, we have

$$P_\sigma(x) = x^4 - A(\sigma)x^3 + (2 + \sigma^{q+1} + \sigma^{-(q+1)} + \sigma^{q-1} + \sigma^{1-q})x^2 - A(\sigma)x + 1.$$

In [14] we had an error in the coefficient for x^2. In fact, $\sigma^{q+1} = \pi_\alpha(h_{1\alpha}^{p^d})$ and $\sigma^{q-1} = \pi_\alpha(h_{2\alpha}^{p^d})$, and $h_{1\alpha}$ and $h_{2\alpha}$ have only two conjugates in H_α, so that

$$\hat{\pi}_\alpha(h_{1\alpha}^{p^d}) = \sigma^{q+1} + \sigma^{-(q+1)}, \qquad \hat{\pi}_\alpha(h_{2\alpha}^{p^d}) = \sigma^{q-1} + \sigma^{1-q}$$

and if we replace p^d by 2^{d+1}, we see that the lemma holds $\pmod 2$.

3.1.3. $G_2(3^k)$. The central characters $\pmod 2$ for $G_2(3^k)$ essentially coincide with the tables for $G_2(q)$, $2, 3 \nmid q$. Namely, for all unipotent $u \neq 1$ in $G_2(3^k)$ we have $\omega_\chi(\hat{u}) \equiv 0 \pmod 2$ for all central characters of nonzero defect, as in the tables in the Appendix. For the remainder of the 2-regular class sums, after relabelling as in [7], these agree exactly with the tables for $G_2(q)$, $2, 3 \nmid q$.

3.2. $q \equiv 1 \pmod 4$.

3.2.1. *The principal block* B_1. Clearly, $X_{11} = 1_G$, X_{12}, X_{13}, X_{14}, X_{15}, X_{16}, X_{17}, $X_{18} \in B_1$.

Since $(\pi_1^\times)^2 = 1$, we have $\pi_1^\times(h_1) = \pm 1$ for $h_1 \in H_1$, so that $\pi_1(h_1) \equiv 1 \pmod 2$. This implies X_{21}, X_{22}, X_{23}, $X_{24} \in B_1$.

If $3 \nmid q$, we verify that X_{31}, X_{32}, $X_{33} \notin B_1$:

(a) If $q \equiv 1 \pmod 3$, we would need $\hat{\pi}_1^+(h_{1\alpha}) \equiv 0 \pmod 2$. However, if $|h_{1a}| = q-1$, then $\pi_1^+(h_{1a}) = \omega$, so that

$$\hat{\pi}_1^+(h_{1a}) = 3(\omega + \omega^2) \equiv 1 \pmod 2.$$

(b) If $q \equiv -1 \pmod 3$, then $\pi_2^+(h_{2b}) = \omega$ for $|h_{2b}| = q+1$, thus $\hat{\pi}_2^+(h_{2b}) \equiv 1 \pmod 2$ and so $X_{33} \notin B_1$.

We also have $\pi_b^+(h_b) = \omega$ for $|h_b| = q^2-1$. Therefore, $\pi_b^+(h_{2b}) = \pi_b^+(h_b^{q-1}) = \omega$ and $\pi_b^+(h_{2b}) = \omega + \omega^2 \equiv 1 \pmod 2$, and so X_{31}, $X_{32} \notin B_1$.

We now look at the exceptional families:

X_1: We want $(i, j) \equiv (0, 0) \pmod 2$ as in [14, §5.13] so that

$$\hat{\pi}_1(h_{1a}^{2^d}) = \hat{\pi}_1(h_{1b}^{2^d}) = 6 \equiv 0 \pmod 2.$$

We have $(2^d-1)^2$ solutions to

$$i = sr, \quad j = tr, \qquad 1 \le s, t \le 2^d - 1.$$

We exclude $2(2^d-1)$ pairs (s, t) such that $3s \equiv t \pmod{2^d}$, $s \equiv t \pmod{2^d}$, but these have a common solution, namely $s = t = 2^d - 1$. We also exclude $2(2^d-1)$ pairs (s, t) that solve $2t \equiv 3s \pmod{2^d}$ or $2s \equiv t \pmod{2^d}$ (one solution s for each $t \neq 2^{d-1}$), so we get

$$(2^d-1)^2 - 2(2^d-1) + 1 - 2(2^d-2) = (2^d-4)(2^d-2),$$

and dividing by 12 to get the number of X_1 yields the result in §3.2.1.

X_{1a}, X'_{1a}, X_{1b}, X'_{1b}: Let $\alpha = a$ or b. All $\pi_1^{\#_\alpha}$ defined by $i^\#$ with $i^\# = sr$, $1 \le s \le 2^d - 1$, give $X_{1\alpha}$, $X'_{1\alpha} \in B_1$, excepting $i^\# = 2^{d-1} \cdot r$, which would be of order 2. Since for $i^\# \neq 2^{d-1} \cdot r$, we have $i^\#$ and $-i^\#$ with different $\pi_1^{\#_\alpha}$ but the same $\hat{\pi}_1^{\#_\alpha}$, we get $\frac{1}{2}(2^d-2)$ characters $X_{1\alpha}$, $X'_{1\alpha}$ in B_1 as in §3.2.1.

X_a, X_b: By Lemma 3.4 in [14], if π_α defines X_α ($\alpha = a$ or b), then

$$X_\alpha \in B_1 \Leftrightarrow \pi_\alpha(h_\alpha^{2^d+1}) = 1.$$

So, if $\pi_\alpha(h_\alpha) = \xi^i$, where $|h_\alpha| = q^2 - 1 = |\xi|$ and $\xi \in \mathbb{C}$, we need $i \equiv 0 \pmod{rr'}$. However, $q+1$, $q-1 \nmid i$, so only *odd* multiples of rr' are possible. This yields $\frac{1}{2} \cdot 2^{d+1} = 2^d$ π_α, and dividing by 4, we get the number of X_α: 2^{d-2}.

Claim. No characters of types X_2, X_{2a}, X'_{2a}, X_{2b}, X'_{2b} are in B_1.

Proof. If $X_2 \in B_1$, then the π_2 defining it must satisfy $\pi_2 \sim (i, j) \equiv (0, 0) \pmod{r'}$. However, $q + 1 = 2r'$, so π_2 is of order 1 or 2, a contradiction.

Now let $\alpha = a$ or b. For $|h_\alpha| = q^2 - 1$ we denote $\pi_\alpha^*(h_\alpha) = \xi^{(q-1)\iota^*} = \sigma$, where $\xi \in \mathbb{C}$ of order $q^2 - 1$ as before. We have

$$\pi_\alpha(h_{2\alpha}) = \pi_\alpha(h_\alpha^{q-1}) = \sigma^{q-1} = \sigma^{-2}.$$

Since $h_{2\alpha}^2$ is of odd order, we have, if π_α^* defines $X_{2\alpha}$, $X'_{2\alpha} \in B_1$,

$$\hat{\pi}_\alpha(h_{2\alpha}^2) = \sigma^{-4} + \sigma^4 \equiv 0 \pmod 2.$$

Hence, $\sigma^{\pm 4} = 1$. However, $\sigma^{q+1} = 1$, so σ^2 is of odd order and therefore $\sigma^2 = 1$, giving $\sigma = \pm 1$. But then, $(\pi_\alpha^*)^2 = 1$, which is not possible. □

3.2.2. *The block* B_3 *if* $q \equiv 1 \pmod 3$. Let B_3 be the block containing X_{31}, X_{32}, X_{33}.

Note. If $|h_{1a}| = q - 1 = |h_{1b}|$, we have

$$\pi_1^+(h_{1a}) = \omega, \qquad \pi_1^+(h_{1b}) = 1,$$

so that

$$\hat{\pi}_1^+(k_3) = 1 + 1 \equiv 0 \pmod 2 \quad \text{as } k_3 = h_{1b}^{(q-1)/3},$$
$$\hat{\pi}_1^+(h_{1b}) = 6 \equiv 0 \pmod 2,$$

but

$$\omega_{3i}(\hat{h}_{1a}) = \hat{\pi}_1^+(h_{1a}) = 3(\omega + \omega^2) \equiv 1 \pmod 2 \quad \text{for } 1 \le i \le 3.$$

We also have (using [14, §5.5]) that if $h_1 \in H_1 - (H_{1a} \cup H_{1b})$, then

$$\hat{\pi}_1(h_1) = 6(\omega + \omega^2) \equiv 0 \pmod 2.$$

We then conclude by inspection of the tables that since $\omega_\chi(\hat{h}_{1a}) \equiv 0 \pmod 2$ for χ equal to a character of one of the types X_2, X_b, X_{2b}, X'_{2b}, these are not contained in B_3.

We use Lemma 5.3 from [14] to deal with X_1, X_{1a}, X'_{1a}, X_{1b}, X'_{1b}:

X_1: $X_1 \in B_3$ if it is defined by $\pi_1 \sim (i, j)$, $(i, j) \equiv (\frac{r}{3}, 0) \pmod r$, gives 2^{2d} pairs

$$i = \frac{r}{3} + sr, \quad j = tr, \qquad 0 \le s, t \le 2^d - 1,$$

excluding solutions to

$$\begin{cases} 1 + 3s \equiv 2t \pmod{2^d}, \\ 1 + 3s \equiv t \pmod{2^d}, \\ t = 0, \end{cases}$$

as in [14, §5.15].

For each s there are 2^d solutions t to the last two congruences. To the first there are no solutions t for odd s, but two solutions t for even s (these are t and $2^{d-1} + t$ for $1 \le t \le 2^{d-1}$). As in [14], $(s, 0)$ solves all three congruences, so we get $(2^{2d} - 3 \cdot 2^d + 2)$ solutions π_1 giving $\frac{1}{6}(2^d - 1)(2^d - 2)$ characters $X_1 \in B_3$.

X_{1a}, X'_{1a}: As in [14, §5.15], we have $2^d - 1$ characters of these types in B_3 defined by

$$\pi_1^{\#_a} \sim (2i^\#, 3i^\#) \equiv \pm\left(\frac{r}{3}, 0\right) \pmod r,$$

where we get

$$i^\# = \frac{2r}{3} + tr, \qquad 0 \le t \le 2^d - 1$$

(excluding $t = \frac{2}{3}(2^d - 1)$ if $2^d \equiv 1 \pmod 3$ and $t = \frac{2}{3}(2^d - 2)$ if $2^d \equiv -1 \pmod 3$).

X_{1b}, $X'_{1b} \notin B_3$ as in [14, §5.15].

X_a, X_{2a}, X'_{2a}: By the note at the beginning of §3.2.2, we need to find π_a defining one of X_{2a}, X'_{2a} such that

$$\begin{cases} \hat{\pi}_a(h_{1a}) \equiv 1 \pmod 2, & |h_{1a}| = r, \\ \hat{\pi}_a(h) \equiv 0 \pmod 2, & h \in H_a \backslash H_{1a} \text{ of odd order.} \end{cases}$$

These conditions hold for π_a^+.

Lemma. *If σ, τ are of odd order and $\sigma^{q^2-1} = \tau^{q^2-1} = 1$, then*

$$\sigma + \sigma^{-1} \equiv \tau + \tau^{-1} \pmod 2 \Leftrightarrow \sigma^{\pm 1} = \tau.$$

Proof. $\Leftarrow$ is obvious, and $\Rightarrow$ follows from the fact that the left-hand side implies that

$$(x - \sigma)(x - \sigma^{-1}) \equiv (x - \tau)(x - \tau^{-1}) \mod 2,$$

so that $\tau \equiv \sigma^{\pm 1} \pmod 2$, giving also equality since they are of odd order. □

Hence, if X_{2a}, $X'_{2a} \in B_3$ are defined by π_a^* and $\pi_a^*(h_a) = \xi^{(q-1)i^*}$, then by the lemma, since $\pi_a^*(h_{1a}) \equiv 1 \pmod 2$, we must have $\pi_a^*(h_{1a}) = \omega^{\pm 1}$; but then

$$\omega^{\pm 1} = \pi_a^*(h_{1a}) = \pi_a^*(h_a^{q+1}) = \xi^{(q-1)i^*(q+1)} = 1,$$

which is a contradiction.

Conclusion. X_{2a}, $X'_{2a} \notin B_3$.

X_a: Again, we need π_a such that on odd-order elements $\pi_a = \pi_a^+$, and by Lemma 3.4 in [14], only such π_a are possible. In other words, for $|h_a| = q^2 - 1$ we need

$$\pi_a(h_a^{2^d+1}) = \omega^{\pm 1},$$

so that, if $\pi_1(h_a) = \xi^i$, we must have

$$i \equiv \pm \tfrac{1}{3} rr' \pmod{rr'}.$$

We exclude all i divisible by $q+1$ or $q-1$, so we need *odd* multiples of $\frac{1}{3}rr'$ that are not divisible by 3. This gives us $\frac{1}{2}(3 \cdot 2^{d+1}) \cdot \frac{2}{3} = 2^d + 1$ different i, and so leaves 2^{d-1} characters X_a in B_3.

3.2.3. *The block B_3 if $q \equiv -1 \pmod 3$.* Let B_3 be the block containing X_{31}, X_{32}. We calculate π_b^+: If $|h_b| = q^2 - 1$, then $\pi_b^+(h_b) = \omega^{\pm 1}$, so that

$$\hat{\pi}_b^+(h_{2b}) = \hat{\pi}_b^+(h_b^{q-1}) = \omega + \omega^2 \equiv 1 \pmod 2,$$

and for $h \notin H_{2b}$ one has $\hat{\pi}_b^+(h) \equiv 0 \pmod 2$.

Hence, we have $X_{33} \in B_3$, since

$$\pi_2^+ \sim \pm(0, \tfrac{1}{3}(q+1)),$$

so that

$$\hat{\pi}_2^+(h_2) = 6(\omega + \omega^2) \equiv 0 \pmod 2, \qquad \hat{\pi}_2^+(h_{2a}) = 6 \equiv 0 \pmod 2,$$
$$\hat{\pi}_2^+(k_3) = \hat{\pi}_2^+(h_{2a}^{\frac{1}{3}(q+1)}) \equiv 0 \pmod 2 \quad \text{and} \quad \hat{\pi}_2^+(h_{2b}) = 3(\omega + \omega^2) \equiv 1 \pmod 2.$$

We exclude all characters χ with $\omega_\chi(\hat{h}_{2b}) \equiv 0 \pmod 2$, namely X_1, $X_{1\alpha}$, $X'_{1\alpha}$ ($\alpha = a$ or b), X_a, X_{2a}, X'_{2a}.

Claim. There holds $X_2 \notin B_3$.

Proof. Otherwise, we would need $\pi_2 \sim \pm(0, \frac{1}{3}\cdot r') \pmod{r'}$. Since π_2 defining X_2 cannot be of the form $(0, k)$ (see [14, §5.8] adapted to π_2), we must have

$$i = r' \quad \text{and} \quad j = \tfrac{1}{3}r', \tfrac{2}{3}r', \tfrac{4}{3}r', \tfrac{5}{3}r'.$$

However, $(r', \frac{1}{3}r')$ is of the form $(3k, k)$ and $(r', \frac{1}{3}r')$ of the form $(3k, 2k)$; $(r', \frac{1}{3}r') \equiv -(r', \frac{2}{3}r') \pmod{r'}$ and $(r', \frac{5}{3}r') \equiv -(r', \frac{1}{3}r') \pmod{r'}$. □

X_{2b}, X'_{2b}: We need $\pi_b^* \equiv \pi_b^+ \pmod{r'}$, so we must have $\pi_b^*(h_b) = \xi^{(q-1)i^*} = \omega^{\pm 1}$. This implies that i^* is an odd multiple of $\frac{1}{3}r'$ (so that $(\pi_b^*)^3 \neq 1$). So we have $i^* = \frac{1}{3}r'$ or $\frac{5}{3}r'$, which yield the same $\hat{\pi}_b^*$. Hence, we get one pair X_{2b}, X'_{2b} in B_3 corresponding to $i^* = \frac{1}{3}r'$.

X_b: Again, we need $\pi_b = \pi_b^+$ on elements of odd order. This yields $\pi_b(h_b) = \xi^i$, $i \equiv \pm\frac{1}{3}rr' \pmod{rr'}$, so taking multiples of $\frac{1}{3}rr'$ not divisible by 3 or by 2^d (so that $q-1 \nmid i$), we get $3 \cdot 2^{d+1} \cdot \frac{2}{3} - 4 = 2^{d+2} - 4$ different i, and so $2^d - 1$ characters X_b.

3.2.4. *The blocks* B_{1a}. These are a combination of the blocks B_{1a} and B_a for $p \neq 2$ (see [14, §§2.2 and 5.16]). We fix $i^\#$, $1 \le i^\# \le r-1$, such that $i^\# \neq \frac{r}{3}, \frac{2r}{3}$. Let B_{1a} be the block containing X_{1a}, X'_{1a}. As in [14, §5.16], $X_{1b}, X'_{1b} \notin B_{1a}$. There are 2^d characters X_{1a}, X'_{1a} in B_{1a} with $i^{\#'} \equiv i^\# \pmod{r}$, and $\frac{1}{2}\cdot 2^d(2^d-1) = 2^{2d-1} - 2^{d-1}$ characters X_1 with $\pi_1 \sim (i, j)$:

$$i = 2i^\# + sr, \quad j = 3i^\# + tr, \qquad 0 \le s, t \le 2^d - 1$$

(excluding 2^d pairs (s, t) with $3s \equiv 2t \pmod{2^d}$, two t for every even s).

We now verify

$$X_2, X_{2a}, X'_{2a}, X_{2b}, X'_{2b}, X_b \notin B_{1a}.$$

From the tables we see that we must calculate the values of the $\pi_1^{\#_a}$ that define $X_{1a}, X'_{1a} \in B_{1a}$: Suppose $\pi_1^{\#_a} \sim (2i^\#, 3i^\#)$; $|\rho| = q-1$. Then

$$\begin{cases} \hat{\pi}_1^{\#_a}(h_{1a}) = \rho^{2i^\#} + \rho^{-2i^\#} + 2(\rho^{i^\#} + \rho^{-i^\#}) \\ \qquad\qquad = \pi_1^{\#_a}(h_{1a}) + \pi_1^{\#_a}(h_{1a})^{-1} \pmod 2, \\ \hat{\pi}_1^{\#_a}(h_{1b}) = 2(\rho^{3i^\#} + \rho^{-3i^\#}) + 1 + 1 \equiv 0 \pmod 2, \end{cases}$$

and using [14, §5.5], we obtain

$$\hat{\pi}_1^{\#_a}(h) \equiv 0 \pmod 2 \quad \text{for } h \in H_1 \backslash (H_{1a} \cup H_{1b}).$$

Since $X_{1a}, X'_{1a} \notin B_1$, we must have $\hat{\pi}_1^{\#_a}(h_{1a}) \not\equiv 0 \pmod 2$.

Clearly, then, $X_b, X_2, X_{2b}, X'_{2b} \notin B_{1a}$.

If π_a^* defines X_{2a}, X'_{2a}, then $\pi_a^*(h_a) = \xi^{(q-1)i^*}$ for some i^*, and $\hat{\pi}_a^*(h_{2a}) = 1 + 1 \equiv 0 \pmod 2$. So $X_{2a}, X'_{2a} \notin B_{1a}$.

X_a: We need π_a such that $\pi_a \equiv \pi_a^\#$ on elements of odd order, where $\pi_a^\#$ is the map

$$\pi_a^\#(h_a) = \xi^{(q+1)i^\#} = \rho^{i^\#},$$

$i^\#$ as in $\pi_1^{\#_a}$ defining $X_{1a}, X'_{1a} \in B_{1a}$.

We indeed have

$$\hat{\pi}_a^{\#}(h_{2a}) \equiv 1 + 1 \equiv 0 \pmod 2,$$
$$\hat{\pi}_a^{\#}(h_a) = 2(\rho^{i^{\#}} + \rho^{-i^{\#}}) \equiv 0 \pmod 2,$$
$$\hat{\pi}_a^{\#}(h_{1a}) = \rho^{2i^{\#}} + \rho^{-2i^{\#}} \equiv \hat{\pi}_1^{\#_a}(h_{1a}) \pmod 2.$$

Since $\xi^{(q+1)^2 i^{\#}} = \pi_a^{\#}(h_{1a})$ and $\xi^{(q+1)^2} = \xi^{2(1+q)} = \rho^2$, we need $\pi_a(h_a) = \xi^i$ such that

$$2^{d+1} i \equiv 2^{d+1}(q+1)i^{\#} \quad (\operatorname{mod}(q^2-1)),$$

or

$$i = (q+1)i^{\#} + srr', \qquad 1 \le s \le 2^{d+1} - 1.$$

Since $q + 1 \nmid i$, we only take odd s and obtain 2^d values of i. As in [14, §4.9], qi is of the above form, but not $-i$, $-qi$ since, if $-i \equiv (q+1)i^{\#}$ $(\operatorname{mod} rr')$, there exist t such that $2i^{\#}(q+1) = trr'$ and $4i^{\#} = tr$, giving $i^{\#} \mid r$, a contradiction. This then yields $\frac{1}{2} \cdot 2^d$ characters X_a.

3.2.5. *The blocks* B_{1b}. As in the previous case, we have a block determined by an $i^{\#}$, $1 \le i^{\#} \le r-1$, containing 2^d characters X_{1b}, X'_{1b} defined by $i^{\#'} \equiv i^{\#}$ $(\operatorname{mod} r)$, $\frac{1}{2} \cdot 2^d(2^d - 1)$ characters X_1 defined by $\pi_1 \sim (i, j)$ such that

$$i = i^{\#} + sr, \quad j = 2i^{\#} + tr, \qquad 0 \le s, t \le 2^d - 1,$$

and $\frac{1}{2} \cdot 2^d$ characters X_b with $\pi_b(h_b) = \xi^i$:

$$i = (q+1)i^{\#} + srr', \qquad 1 \le s \le 2^{d+1} - 1, \ s \text{ odd}.$$

Calculating, we see that

$$\begin{cases} \hat{\pi}_1^{\#_b}(h_{1a}) \equiv 0 \pmod 2, \\ \hat{\pi}_1^{\#_b}(h_{1b}) \equiv \pi_1^{\#_b}(h_{1b}) + \pi_1^{\#_b}(h_{1b})^{-1} \pmod 2, \\ \hat{\pi}_1^{\#_b}(h) \equiv 0 \pmod 2 \quad \text{for } h \in H_1 \backslash (H_{1a} \cup H_{1b}). \end{cases}$$

Therefore, since X_{1b}, $X'_{1b} \notin B_1$, we have $\hat{\pi}_1^{\#_b}(h_{1b}) \not\equiv 0 \pmod 2$, giving

$$X_2, X_a, X_{2a}, X'_{2a}, X_{1a}, X'_{1a} \notin B_{1b},$$

and since $\hat{\pi}_b^*(h_{1b}) \equiv 0 \pmod 2$ for any $\pi^*(h_b) = \xi^{(q-1)i^*}$, we also have X_{2b}, $X'_{2b} \notin B_{1b}$.

3.2.6. *The blocks* B_{2a}. These are a combination of the blocks B_{2a} and B_a in [14, §§2.1 and 2.4]. We fix i^*, $1 \le i^* \le r' - 1$, and let B_{2a} be the block containing X_{2a}, X'_{2a} defined by i^*. Since X_{2a}, $X'_{2a} \notin B_1$ and $\hat{\pi}_a^*(h_a) \equiv \hat{\pi}_a^*(h_{1a}) \equiv 0 \pmod 2$, we have $\hat{\pi}_a^*(h_{2a}) \not\equiv 0 \pmod 2$ for h_{2a} of order $q+1$. Hence (by the tables), the only other characters in the block must be of types X_2, X_a, X_{2a}, X'_{2a}.

X_{2a}, X'_{2a}: If $i^{\#'}$ defines another pair X_{2a}, $X'_{2a} \in B_{2a}$, then we have

$$(q-1)2^{d+1} i^* \equiv (q-1)2^{d+1} i^{*'} \ (\operatorname{mod}(q^2-1)) \quad \text{or} \quad i^{\#'} \equiv i^{\#} \ (\operatorname{mod} r').$$

This yields two solutions: $i^{\#'} = i^*$, $i^* + r'$. So we have one other pair X_{2a}, X'_{2a} in the block corresponding to $i^* + r'$.

X_2: We need $\pi_2 \equiv \pi_2^{*_a}$ on elements of odd order, where $\pi_2^{*_a} \sim (2i^*, i^*)$, since

$$\begin{cases} \hat{\pi}_2^{*_a}(h_{2a}) \equiv \pi_2^{*_a}(h_{2a}) + \pi_2^{*_a}(h_{2a})^{-1} \equiv \hat{\pi}_a^*(h_{2a}) \pmod 2, \\ \hat{\pi}_2^{*_a}(h_{2b}) \equiv 0 \pmod 2, \\ \hat{\pi}_2^{*_a}(h) \equiv 0 \pmod 2 \quad \text{for } h \in H_2 \backslash (H_{2a} \cup H_{2b}). \end{cases}$$

So we need $\pi_2 \sim (i, j)$: $i = 2i^* + sr'$, $j = i^* + tr'$, $0 \leq s, t \leq 1$. The pair $s = 0, t = 1$ yields $(2i^*, i^* + r')$, which we exclude, as it is of the form $(2k, k) \pmod{(q+1)}$; the case $s = t = 0$ is $(2i^*, i^*)$, again of course excluded. Since (i, j) and $(i - j, j)$ give the same $\hat{\pi}_2$, the two pairs $(2i^* + r', r')$ and $(2i^* + r', i^* + r')$ give *one* $X_2 \in B_{2a}$.

X_a: We need $\pi_a(h_a) = \xi^i$ such that

$$2^{d+1} i \equiv 2^{d+1} i^*(q-1) \pmod{(q^2-1)},$$

or $i = i^*(q-1) + srr'$, where $1 \leq s \leq 2^{d+1} - 1$ and $s \neq 2^d$. This gives $2^{d+1} - 2$ solutions and $2^d - 1$ characters X_a.

3.2.7. *The blocks B_{2b}.* As in the previous case, for a fixed i^*, $1 \leq i^* \leq r' - 1$, $i^* \neq \frac{r'}{3}, \frac{2r'}{3}$, we have two pairs X_{2b}, X'_{2b} corresponding to i^* and $i^* + r'$. Also one $X_2 \in B_{2b}$ with $\pi_2 \sim (3i^* + r', 2i^* + r')$ or $(3i^*, 2i^* + r')$. (The pair $(3i^* + r', 2i^*)$ is excluded, as it is of type $(3k, 2k)$.) The above pair yields the same X_2, since (i, j) and $(3j - i, j)$ give the same $\hat{\pi}_2$.

We have $2^d - 1$ characters X_b corresponding to $i = i^*(q-1) + srr'$, as above.

3.2.8. There remain only those X_1, X_2, X_a, X_b not in any of the above blocks.

Lemma. *Let $\alpha = a$ or b, and $\pi_\alpha(h_\alpha) = \xi^i$, where π_α defines X_α and i is a multiple of r or of r'. Then X_α is in B_1, B_3 or in one of the blocks $B_{1\alpha}$, $B_{2\alpha}$.*

Proof. First let $i = tr$, $t \geq 1$. If $r' \mid t$ or $i \equiv \pm\frac{1}{3}rr'$, then $X_\alpha \in B_1 \cup B_3$. Otherwise, we show $i = (2^d i^* + sr')r$ for some i^*, where $1 \leq i^* < q + 1$ and $1 \leq s < 2^{d+1}$, which gives $X_\alpha \in B_{2\alpha}$ for the appropriate block $B_{2\alpha}$ defined by i^*. Since $(2^d, r') = 1$, we can set $i^* \equiv 1/2^d \cdot t \pmod{r'}$ (giving $1 \leq i^* < r'$). Then $t = 2^d i^* + mr'$ for some m. If $m > 0$ we are done. Otherwise, we look at $(q^2 - 1) - i$, which yields the same $\hat{\pi}_\alpha$ (and so the same X_α):

$$(q^2 - 1) - i = [(2^{d+1} - 2^d - m)r' + 2^d(r' - i^*)]r.$$

Since now $m < 0$, we must have, because of $t \geq 1$, that $m < -2^d$, so $1 \leq 2^{d+1} - 2^d - m < 2^{d+1}$ as required, and X_α is in the $B_{2\alpha}$ defined by $r' - i^*$.

Now take $i = tr'$. If $i \equiv \pm\frac{1}{3}rr' \pmod{rr'}$ or if $rr' \mid i$, then $X_\alpha \in B_1 \cup B_3$. Otherwise, we show $i = (2i^\# + sr)r'$ for some $i^\#$, $1 \leq i^\# \leq q - 1$, $1 \leq s < 2^{d+1}$, so that $X_\alpha \in B_{1\alpha}$ for the block $B_{1\alpha}$ defined by $i^\#$. Set $i' \equiv t \pmod r$, $1 \leq i^* \leq r$; then

$$t = i' + sr \quad \text{for } s \geq 0.$$

If i' is even, we are done, as $i' = 2i^\#$ and $i^\#$ defines the block $B_{1\alpha}$ to which X_α belongs. If i' is odd, then if $s > 0$ we have $i' + r$ even and $t = (i' + r) + (s - 1)r$

of the appropriate form, taking $i^{\#} = \frac{1}{2}(i'+sr)$. If $s = 0$ we look at $(q^2-1)-i = (2^{d+1}r - t)r'$, and then $t' = 2^{d+1}r - t > r$ can be written in the above form; t yields the same $\hat{\pi}_\alpha$. □

3.2.9. *The blocks* B_{X_1}. Let $X_1(\pi_1) \notin B_1 \cup B_3$ and $X_1 \notin B_{1a} \cup B_{1b}$ for all blocks of these types. Denote by B_{X_1} the block containing $X_1(\pi_1)$. Assume $\pi_1 \sim (i, j)$.

Claim. There holds $X_2, X_a, X_b \notin B_{X_1}$.

Proof. If $X_2 \in B_{X_1}$, we would need $\hat{\pi}_1(h) \equiv 0 \pmod 2$ for all $h \in H_1$ giving $X_1 \in B_1$. If $X_\alpha \in B_{X_1}$ ($\alpha = a$ or b), we would have $\hat{\pi}_\alpha(h_{2\alpha}) \equiv 0 \pmod 2$ for π_α defining X_α. This implies $\xi^{2(q-1)i} = 1$, $\pi_\alpha(h_\alpha) = \xi^i$, and so $r' \mid i$. By Lemma 3.2.8, this means $X_\alpha \notin B_{X_1}$. □

As in [14, §5.18], we have a total of 2^{2d} characters of type X_1 in B_{X_1} corresponding to pairs (i', j') such that $(i', j') \equiv (i, j) \pmod r$. Counting the blocks B_{X_1}, we exclude all (i, j) such that

$$\begin{aligned} 2i &\equiv j \pmod r, & 3i &\equiv 2j \pmod r, \\ i &\equiv j \pmod r, & 3i &\equiv j \pmod r. \end{aligned}$$

If $q \not\equiv 1 \pmod 3$, then each congruence has $r - 1$ solutions, as r is odd and $3 \nmid r$, and none of the solutions occurs more than once. This gives $(r-1)^2 - 4(r-1) = (r-1)(r-5)$ characters π_1, and so $\frac{1}{12}(r-1)(r-5)$ blocks. If $q \equiv 1 \pmod 3$, then $3 \mid r$, so we get $r - 1$ solutions to $2i \equiv j$, $i \equiv j$, and $r - 3$ solutions to $3i \equiv j$, $3i \equiv 2j$ (as in [14, §5.18]), and thus get $(r-1)^2 - 2(r-1) - 2(r-3) = (r-3)^2$ characters π_1 and $\frac{1}{12}(r-3)^2$ blocks.

3.2.10. *The blocks* B_{X_2}. Let B_{X_2} be the block containing $X_2(\pi_2)$ for some $X_2 \notin B_1 \cup B_3 \cup B_{2a} \cup B_{2b}$. Assume $\pi_2 \sim (i, j)$. As in [14, §2.4], B_{X_2} contains four characters of type X_2 corresponding to the pairs

$$(i, j + r'), \quad (i + r', j), \quad (i + r', j + r'), \quad \text{and} \quad (i, j).$$

We have $X_\alpha \notin B_{X_2}$ for $\alpha = a$ and b, since otherwise, for π_α defining X_α, we would have $\pi_\alpha(h_{1\alpha}) \equiv 0 \pmod 2$, implying $\pi_\alpha(h_\alpha) = \xi^{ir}$ for some r. This contradicts Lemma 3.2.8.

3.2.11. *The blocks* B_{X_α}, $\alpha = a$ *or* b. Let $\pi_\alpha(h_\alpha) = \xi^i$ define $X_\alpha(\pi_\alpha)$, where i is not a multiple of r or r', and $1 \le i < rr'$. Denote by B_{X_α} the block containing X_α.

The block B_{X_α} contains all X_α defined by $i' = i+srr'$, $1 \le s \le 2^{d+1}$, giving 2^{d+1} characters X_α. (These all give different $\hat{\pi}_\alpha$ as $-i$, qi, $-qi$ are not of this form. For instance, if we had $i \equiv qi \pmod{rr'}$, then $(q-1)i \equiv 0 \pmod{rr'}$, which implies $i \equiv 0 \pmod{r'}$, a contradiction.)

Clearly, $X_b \notin B_{X_a}$ for all X_b, since otherwise, if π_b defines X_b, we would have $\pi_b(h) \equiv 0 \pmod 2$ for all $1 \ne h \in H_b$, giving $X_b \in B_1$. Similarly $X_a \notin B_{X_b}$ for all X_a.

Number of blocks of this type: $\frac{1}{4}(rr' - r - r' + 1) = \frac{1}{8}(r-1)(q-1)$.

3.2.12. $q \equiv -1 \pmod 4$. The proofs here are analogous to those for $q \equiv 1 \pmod 4$. We merely exchange $q+1$ for $q-1$ and also the subscripts 1 and 2, a and b, and superscripts # and $*$.

4. Proofs: decomposition matrices

4.1. **Some scalar products.** The following tables list some scalar products between characters of G. We only give those scalar products we shall need in our proofs and which have not already been given in Appendix A of [7]. As always, missing entries are 0.

(a) $q \equiv 0 \pmod 3$ (Table (q)).

TABLE (q)

Char.	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	X_{17}	X_{18}
$X_{19} \otimes X_{13}$		$q^2/9$		$q/3$				
$X_{19} \otimes X_{14}$		$q^2/9$	$q/3$					
$X_{19} \otimes X_{19}$		$q^2/9$				$q/3$		$q/3$
$X_{19} \otimes \overline{X}_{19}$	1	$q^2/9$	$q/3$	$q/3$				
$(X_3 + X_6) \otimes X_{22}$		$2(q^2+1)$	$2q/3$	$2q/3$	q	$q/3$	q	$q/3$

(b) $q \equiv \varepsilon \pmod 3$, $\varepsilon = 1, -1$ (Table (r)).

TABLE (r)

Char.	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	X_{17}	X_{18}
$X_{19} \otimes X_{13}$		$(q-\varepsilon)^2/9$		$(q-\varepsilon)/3$				
$X_{19} \otimes X_{14}$		$(q+2\varepsilon)(q-\varepsilon)/9$	$(q-\varepsilon)/3$					
$X_{19} \otimes X_{19}$		$(q+2\varepsilon)(q-\varepsilon)/9$				$(q-\varepsilon)/3$		$(q-\varepsilon)/3$
$X_{19} \otimes \overline{X}_{19}$	1	$(q+2\varepsilon)(q-\varepsilon)/9$	$(q-\varepsilon)/3$	$(q+2\varepsilon)/3$				
$X_3 \otimes X_{32}$		$q-\varepsilon$			1		1	

4.2. **The proof for the principal block B_1.** We have the following relations on 2-regular classes:

$$\begin{aligned} X_{16} &= X_{15} - X_{17} + X_{18}, \\ X_{21} &= X_{12} + X_{15} - X_{17}, \\ X_{22} &= X_{11} + X_{15} - X_{17}, \\ X_{23} &= X_{14} + X_{15} + X_{18}, \\ X_{24} &= X_{13} + X_{15} + X_{18}. \end{aligned}$$

If $\delta = -1$, we have furthermore

$$\begin{aligned}
X_{2a} &= X_{12} - X_{13} - X_{17} - X_{18}, \\
X'_{2a} &= -X_{11} + X_{14} + X_{17} + X_{18}, \\
X_{2b} &= X_{12} - X_{14} - X_{17} - X_{18}, \\
X'_{2b} &= -X_{11} + X_{13} + X_{17} + X_{18}, \\
X_a &= -X_{11} + X_{12} - X_{13} + X_{14} = X_{2a} + X'_{2a}, \\
X_b &= -X_{11} + X_{12} + X_{13} - X_{14} = X_{2b} + X'_{2b}, \\
X_2 &= X_{11} + X_{12} - X_{13} - X_{14} - 2 \cdot X_{17} - 2 \cdot X_{18} \\
&= X_{2a} - X'_{2a} = X_{2b} - X'_{2b}.
\end{aligned}$$

If $\delta = 1$, these are replaced by

$$\begin{aligned}
X_{1a} &= X_{12} + X_{14} + 2 \cdot X_{15} - X_{17} + X_{18}, \\
X'_{1a} &= X_{11} + X_{13} + 2 \cdot X_{15} - X_{17} + X_{18}, \\
X_{1b} &= X_{12} + X_{13} + 2 \cdot X_{15} - X_{17} + X_{18}, \\
X'_{1b} &= X_{11} + X_{14} + 2 \cdot X_{15} - X_{17} + X_{18}, \\
X_a &= -X_{11} + X_{12} - X_{13} + X_{14} = X_{1a} - X'_{1a}, \\
X_b &= -X_{11} + X_{12} + X_{13} - X_{14} = X_{1b} - X'_{1b}, \\
X_1 &= X_{11} + X_{12} + X_{13} + X_{14} + 4 \cdot X_{15} - 2 \cdot X_{17} + 2 \cdot X_{18} \\
&= X_{1a} + X'_{1a} = X_{1b} + X'_{1b}.
\end{aligned}$$

Since X_{11}, X_{12}, X_{13}, X_{15}, X_{14}, X_{17}, and X_{18} are linearly independent on 2-regular classes, they form a basic set by Lemma 4 of [7]. Table (s) gives a list of scalar products, where u, v, w, x, y, and z are nonnegative integers. The projectives originate from Table (t) (see next page).

TABLE (s)

Char.	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ'_7	Φ_8	Φ_9
X_{11}	1					1			1	1
X_{17}							1	q		
X_{18}		1						$q/3$		
X_{13}	1		1			u		$2q/3$	q	1
X_{14}	1			1		v		$2q/3$	1	q
X_{15}	2						1	q	$q+1$	$q+1$
X_{12}	1	x	y	z	1	w	$q-1$	$2(q^2+1)$	q	q

TABLE (t)

Char.	Origin	Remarks
Φ_1	1_U^G	
Φ_2	$X_{19} \otimes X_{19}$	X_{19} defect 0
Φ_3	$X_{19} \otimes X_{14}$	X_{19} defect 0
Φ_4	$X_{19} \otimes X_{13}$	X_{19} defect 0
Φ_5	G-G	
Φ_6	$X_{19} \otimes \overline{X}_{19}$	X_{19} defect 0
Φ_7	$X_3 \otimes X_{32}$	only if $3 \nmid q$
Φ_7'	$(X_3 + X_6) \otimes X_{22}$	if $3 \mid q$, X_3, X_6 defect 0
Φ_8	$1_{U_{\{b\}}}^G$	[7, Appendix A]
Φ_9	$1_{U_{\{a\}}}^G$	[7, Appendix A]

Now Φ_6 shows that X_{15} is not a constituent of Φ_{11}, the projective indecomposable corresponding to the trivial character X_{11}. On the other hand, by Fong's lemma (Theorem 6.3.86 of [10]), X_{13}, X_{14}, and X_{12} are contained in Φ_{11}. Hence, there is just one way Φ_1 can break up into projectives: $\Phi_1 = \Phi_{11} + 2 \cdot \Phi_{15}$.

Let Φ denote the projective indecomposable character contained in Φ_7, resp. Φ_7', which has nonzero scalar product with X_{17}. Since the decomposition matrix has only 1's as elementary divisors, X_{17} is contained just once in Φ. Thus, we get the set of projectives given in Table (u). We observe that Φ_{11}, Φ, $\Phi_2, \ldots, \Phi_5$, and Φ_{15} are a basis for the set of projectives of the principal block.

TABLE (u)

Char.	Φ_{11}	Φ	Φ_2	Φ_3	Φ_4	Φ_{15}	Φ_5	Φ_7	Φ_7'	Φ_8	Φ_9
X_{11}	1									1	1
X_{17}		1						1	q		
X_{18}		a	1						$q/3$		
X_{13}	1	b		1					$2q/3$	q	1
X_{14}	1	c			1				$2q/3$	1	q
X_{15}		d				1		1	q	$q+1$	$q+1$
X_{12}	1	e	x	y	z		1	$q-1$	$2(q^2+1)$	q	q

Now Φ is certainly not contained in Φ_8 or Φ_9. It follows that Φ_{15} is contained $(q+1)$ times in each of these. Furthermore, Φ_{11} is contained once in each of Φ_8 and Φ_9. Subtracting these from Φ_8, respectively Φ_9, leaves multiples of projectives with multiplicity $(q-1)$. We thus get the set of projectives given in Table (v).

TABLE (v)

Char.	Φ_{11}	Φ	Φ_2	Φ_{13}	Φ_{14}	Φ_{15}	Φ_{12}	Φ_7	Φ_7'
X_{11}	1								
X_{17}		1						1	q
X_{18}		a	1						$q/3$
X_{13}	1	b		1					$2q/3$
X_{14}	1	c			1				$2q/3$
X_{15}		d				1		1	q
X_{12}	1	e	x	1	1		1	$q-1$	$2(q^2+1)$

If $3 \mid q$, then Φ is contained q times in Φ_7'. It follows that $a = b = c = 0$, that $d \leq 1$, and that $e \leq 2q$. This is of course trivially true in case $3 \nmid q$, by considering Φ_7. Since $X_{16} = X_{15} - X_{17} + X_{18}$ on 2-regular classes, we must have that $d = 1$. The missing entries of the decomposition matrix are now filled in, using the relations given above. The tensor product $X_{19} \otimes X_{19}$ shows that $x \leq (q+2)/3$. If 3 does not divide q, use Φ_7 to get the bound $e \leq q$, and, with $x \leq q$, the lower bound for $\phi_{12}(1)$. If $3 \mid q$ and $q > 3$, we get from Φ_7' that $3e + x \leq 6q$. This yields the lower bound for $\phi_{12}(1)$ in this case, and completes the proof for the principal block.

4.3. **The block B_3.**

4.3.1. *The case* $q \equiv -1 \pmod 3$. Here we have the following relations:

$$\begin{aligned} X_{2b} &= X_{31} - X_{33}, \\ X'_{2b} &= X_{32} + X_{33}, \\ X_b &= X_{31} + X_{32} = X_{2b} + X'_{2b}, \\ X_2 &= X_{31} - X_{32} - 2 \cdot X_{33} = X_{2b} - X'_{2b}. \end{aligned}$$

Of course, the last relation only makes sense in case $4 \mid q+1$, since otherwise, X_2 is not contained in B_3. Since X_{31}, X_{32}, X_{33} are linearly independent on 2-regular classes, they form a basic set by Lemma 4 of [7]. Table (w) gives a table of scalar products with projective characters, with the projectives originating from Table (x) (see next page).

TABLE (w)

char.	Φ_1	Φ_2	Φ_3
X_{32}	1		
X_{33}		1	
X_{31}	1	γ	1

TABLE (x)

Char.	Origin	Remarks
Φ_1	$1_{U\{b\}}^G$	[7, Appendix A]
Φ_2	$X_{19} \otimes X_{14}$	X_{19} defect 0
Φ_3	G-G	

These projectives show that X_{32} and X_{33} are irreducible modulo 2. If $4 \mid q+1$, then X_2 is in the block, and the last relation shows that X_{32} must be a modular constituent of X_{33}. The tensor product $X_{14} \otimes X_{19}$ yields the desired bound for γ, and the proof is complete in this case.

If $4 \mid q-1$, then the defect group of B_3 is a Sylow 2-subgroup of $SU_3(q)$. It is semidihedral of order 2^{d+2}. The decomposition matrices given in [6, §11] now show that $\gamma = 1$ and that X_{32} is a modular constituent of X_{31}.

4.3.2. *The case* $q \equiv 1 \pmod 3$. Here we have the following relations:

$$\begin{aligned} X_{1a} &= X_{31} + X_{33}, \\ X'_{1a} &= X_{32} + X_{33}, \\ X_a &= X_{31} - X_{32} = X_{1a} - X'_{1a}, \\ X_1 &= X_{31} + X_{32} + 2 \cdot X_{33} = X_{1a} + X'_{1a}. \end{aligned}$$

Of course, the last relation only makes sense if $4 \mid q-1$, since otherwise X_1 is not contained in B_3. Since X_{31}, X_{32}, X_{33} are linearly independent on 2-regular classes, they form a basic set by Lemma 4 of [7]. Table (y) gives a table of scalar products with projective characters, with the projectives originating from Table (z).

TABLE (y)

char.	Φ_1	Φ_2	Φ_3
X_{32}	1		
X_{33}	2	1	
X_{31}	1	x	1

TABLE (z)

Char.	Origin	Remarks
Φ_1	$1_{U\{b\}}^G$	[7, Appendix A]
Φ_2	$X_{19} \otimes X_{14}$	X_{19} defect 0
Φ_3	G-G	

The defect group of B_3 is contained in $SL_3(q)$. If $4 \mid q+1$, this defect group is semidihedral of order 2^{d+2}. In [6, §11], Erdmann enumerates the possible decomposition matrices of blocks with semidihedral defect group and three irreducible Brauer characters. From those results it follows that the indecomposable Φ_3 is contained x times in Φ_2, and that $(\Phi_2 - x\Phi_3)$ is contained twice in Φ_1.

Now let $4 \mid q-1$. We shall show that X_{33} remains irreducible on reduction modulo 2. This will complete the proof, since the relation for X_a then shows that there is only one way Φ_1 can break up into indecomposables. Let $H = \mathscr{H}_1$ be the split maximal torus, B a Borel subgroup containing H, and $N = N_G(H)$ the Cartan subgroup of G. Let λ be one of the two ordinary irreducible characters of H whose Harish-Chandra induction is $X_{31} + X_{32} + 2X_{33}$. Let S denote the Sylow 2-subgroup of H. Since $4 \mid q-1$, we have $C_G(S) = H$. Since $N_G(S)$ normalizes $C_G(S)$, we certainly have $N_G(S) = N$. Also, $N_B(S) = H$.

Next we choose a splitting 2-modular system (K, R, k). As usual, R denotes a rank-1 complete discrete valuation ring with field of fractions K and residue class field k of characteristic 2. Let l be an RH-lattice with character λ, and let L denote the inflation of l to B. By [9, Theorem 2.18], l is extendible to its inertia subgroup T in N. Since T/H is isomorphic to a symmetric group on three letters, l^T is a direct sum of three indecomposable modules. Each of these has dimension 2 and vertex S, and two of these are isomorphic. By Green's theorem, l^N is the direct sum of three indecomposable modules of dimension 4 and vertex S, two of which are isomorphic. Furthermore, every indecomposable direct summand of l^N is self-dual.

We now apply Burry's generalized version of Green correspondence (see [1, Theorem 4.2(a)]) to L^G. It states that the number of direct summands of L^G with vertex S (counting multiplicities) is the same as the number of direct summands with vertex S in $(L_{N_B(S)})^{N_G(S)} = (L_H)^N = l^N$. Hence, $L^G = X \oplus Y \oplus Y$, where X has character $X_{31} + X_{32}$ and Y has character X_{33}.

Now Y is certainly self-dual, and so is $\overline{Y}$, the reduction of Y modulo 2. Since L is a trivial source module (being a direct summand of l^B), $\overline{Y}$ has trivial source, and so all its endomorphisms are liftable (see [11, Theorem II 12.4]). Every irreducible kG-module in B_3 is self-dual, since it has a real-valued Brauer character. From this, and the fact that $\overline{Y}$ is self-dual with a 1-dimensional endomorphism ring, it follows that $\overline{Y}$ is irreducible. This completes the proof.

4.4 **The blocks B_{1a} and B_{1b}.** The proof for B_{1b} is exactly the same as that for B_{1a}, so we only give the latter. The characters of type X'_{1a} which lie in the block have the same restriction to the 2-regular classes. The same is true for the characters of type X_{1a}. Furthermore, we have the following relations on 2-regular classes: $X_1 = X_{1a} + X'_{1a}$ and $X_a = X_{1a} - X'_{1a}$. We have projectives

Char.	Φ_1	Φ_2
X'_{1a}	1	
X_{1a}	1	1

Here, Φ_1 originates from $\mathbf{1}_{U_{\{b\}}}^G$ (divided by $3q+3$ (see [7, p. 349])) and Φ_2 from the Gelfand-Graev character. The relations show that each of these projectives must be indecomposable, and we are finished with the proof.

4.5. **The blocks B_{2a} and B_{2b}.** The proofs here are similar to the ones above. In place of $\mathbf{1}_{U_{\{b\}}}^G$ we have to take $\mathbf{1}_{U_{\{a\}}}^G$.

APPENDIX

TABLE 1 (a)

$q \equiv 1 \pmod 4$

$G_2(q) =$ Central char. on $2'$-elements (mod 2)

$q - 1 = 2^d \cdot r$, $q + 1 = 2r'$, r and r' odd

	defect.	$2d+2$	$2d+2$	$2d+1$	3	$2d+2$	$2d+1$	$d+2$	2
	$2, 3 \nmid q$ $G_2(q)$	ω_{11} ω_{12}	ω_{13} ω_{14}	ω_{15} ω_{16}	ω_{17} ω_{18}	ω_{21}, ω_{22} ω_{23}, ω_{24}	ω_{31}, ω_{32} ω_{33} $\varepsilon = 1$	ω_{31}, ω_{32} $\varepsilon = -1$	ω_{33} $\varepsilon = -1$
	1	1	1	1	1	1	1	1	1
	$\hat{u}_1$	0							
	$\hat{u}_2$	0							
	$\hat{u}_3$	0							
	$\hat{u}_4$	0							
	$\hat{u}_5$	0							
	$\hat{u}_6$	0							
	$\hat{k}_3$	0	$\hat{I}_1(k_3) \equiv 0$	$\hat{I}_1(k_3) \equiv 0$		$\hat{\pi}_1^{\times}(k_3) \equiv 0$	$\hat{\pi}_1^{+}(k_3) \equiv 0$		
$\varepsilon = 1$	$\hat{k}_{31}$	0							
	$\hat{k}_{32}, k_{3,3,\iota}$	0							
	$\hat{k}_3$	0			$\hat{I}_2(k_3) \equiv 0$				$\pi_2^{+}(k_3)$
$\varepsilon = -1$	$\hat{k}_{31}$	0							
	$\hat{k}_{32}, k_{3,3,\iota}$	0							
	$\hat{h}_{1a}$	0	$\hat{I}_1(h_{1a}) \equiv 0$	$\hat{I}_1(h_{1a}) \equiv 0$		$\hat{\pi}_1^{\times}(h_{1a}) \equiv 0$	$\hat{\pi}_1^{+}(h_{1a})$		
	$\hat{h}_{1a,1}$	0							
	$\hat{h}_{1b}$	0	$\hat{I}_1(h_{1b}) \equiv 0$	$\hat{I}_1(h_{1b}) \equiv 0$		$\hat{\pi}_1^{\times}(h_{1b}) \equiv 0$	$\hat{\pi}_1^{+}(h_{1b}) = 0$	$\hat{\pi}_b(h_{1b}) \equiv 0$	
	$\hat{h}_{1b,1}$	0							
	$\hat{h}_{2a}$	0			$\hat{I}_2(h_{2a}) \equiv 0$				$\hat{\pi}_2^{+}(h_{2a}) \equiv 0$
	$\hat{h}_{2a,1}$	0							
	$\hat{h}_{2b}$	0			$\hat{I}_2(h_{2b}) \equiv 0$			$\hat{\pi}_b^{+}(h_{2b})$	$\hat{\pi}_2^{+}(h_{2b})$
	$\hat{h}_{2b,1}$	0							
	$\hat{h}_1$	0	$\hat{I}_1(h_1) \equiv 0$	$\hat{I}_1(h_1) \equiv 0$		$\hat{\pi}_1^{\times}(h_1) \equiv 0$	$\hat{\pi}_1^{+}(h_1) \equiv 0$		
	$\hat{h}_2$	0			$\hat{I}_2(h_2) \equiv 0$				$\hat{\pi}_2^{+}(h_2) \equiv 0$
	$\hat{h}_a$	0							
	$\hat{h}_b$	0						$\hat{\pi}_b^{+}(h_b) \equiv 0$	
	$\hat{h}_3$	0							
	$\hat{h}_6$	0	0	0	0	0	0	0	0

TABLE 1 (b)
$q \equiv 1 \pmod 4$
$G_2(q) =$ Central char. on $2'$-elements (mod 2)
$q-1 = 2^d \cdot r,\ \ q+1 = 2r',\ \ r$ and r' odd

	defect:	$2d$	2	$d+1$	$d+2$	$2d+1$	$d+1$	$d+2$	$2d+1$
	$2, 3 \nmid q$	ω_1	ω_2	ω_a	$\omega_{2a}, \omega'_{2a}$	$\omega_{1a}, \omega'_{1a}$	ω_b	$\omega_{2b}, \omega'_{2b}$	$\omega_{1b}, \omega'_{1b}$
	1	1	1	1	1	1	1	1	1
	$\hat{u}_1$								
	$\hat{u}_2$								
	$\hat{u}_3$								
	$\hat{u}_4$								
	$\hat{u}_5$								
	$\hat{u}_6$								
	$\hat{k}_3$	$\hat{\pi}_1(k_3)$				$\hat{\pi}_1^{\#a}(k_3)$	$\hat{\pi}_b(k_3)$	$\hat{\pi}_b^*(k_3)$	$\hat{\pi}_1^{\#b}(k_3)$
$\varepsilon = 1$	$\hat{k}_{31}$								
	$\hat{k}_{32}, k_{3,3,t}$								
	$\hat{k}_3$		$\hat{\pi}_2(k_3)$	$\hat{\pi}_a(k_3)$	$\hat{\pi}_a^*(k_3)$				
$\varepsilon = -1$	$\hat{k}_{31}$								
	$\hat{k}_{32}, k_{3,3,t}$								
	$\hat{h}_{1a}$	$\hat{\pi}_1(h_{1a})$		$\hat{\pi}_a(h_{1a})$	$\hat{\pi}_a^*(h_{1a})$	$\hat{\pi}_1^{\#a}(h_{1a})$			$\hat{\pi}_1^{\#b}(h_{1a}) \equiv 0$
	$\hat{h}_{1a,1}$								
	$\hat{h}_{1b}$	$\hat{\pi}_1(h_{1b})$				$\hat{\pi}_1^{\#a}(h_{1b}) \equiv 0$	$\hat{\pi}_b(h_{1b})$	$\hat{\pi}_b^*(h_{1b})$	$\hat{\pi}_1^{\#b}(h_{1b})$
	$\hat{h}_{1b,1}$								
	$\hat{h}_{2a}$		$\hat{\pi}_2(h_{2a})$	$\hat{\pi}_a(h_{2a})$	$\hat{\pi}_a^*(h_{2a})$				
	$\hat{h}_{2a,1}$								
	$\hat{h}_{2b}$		$\hat{\pi}_2(h_{2b})$				$\hat{\pi}_b(h_{2b})$	$\hat{\pi}_b^*(h_{2b})$	
	$\hat{h}_{2b,1}$								
	$\hat{h}_1$	$\hat{\pi}_1(h_1)$				$\hat{\pi}_1^{\#a}(h_1)$			$\hat{\pi}_1^{\#b}(h_1)$
	$\hat{h}_2$		$\hat{\pi}_2(h_2)$						
	$\hat{h}_a$			$\hat{\pi}_a(h_a)$	$\hat{\pi}_a^*(h_a)$				
	$\hat{h}_b$						$\hat{\pi}_b(h_b)$	$\hat{\pi}_b^*(h_b)$	
	$\hat{h}_3$								
	$\hat{h}_6$	0	0	0	0	0	0	0	0

TABLE 2 (a)
$q \equiv -1 \pmod 4$
$G_2(q)$ = Central char. on $2'$-elements (mod 2)
$q-1=2r'$, $q+1=2^d \cdot r$, r and r' odd

	defect:	$2d+2$	$2d+2$	$2d+1$	3	$2d+2$	$2d+1$	$d+2$	2
	$2, 3 \nmid q$ $G_2(q)$	ω_{11} ω_{12}	ω_{13} ω_{14}	ω_{17} ω_{18}	ω_{15} ω_{16}	ω_{21}, ω_{22} ω_{23}, ω_{24}	ω_{31}, ω_{32} $d(X_{33}) = 2d$ ω_{33} $\varepsilon = -1$	ω_{31}, ω_{32} $\varepsilon = 1$	ω_{33} $\varepsilon = 1$
	1	1	1	1	1	1	1	1	1
	$\hat{u}_1$	0							
	$\hat{u}_2$	0							
	$\hat{u}_3$	0							
	$\hat{u}_4$	0							
	$\hat{u}_5$	0							
	$\hat{u}_6$	0							
	$\hat{k}_3$	0			$\widehat{I}_1(k_3) \equiv 0$				$\hat{\pi}_1^+(k_3) \equiv 0$
$\varepsilon = 1$	$\hat{k}_{31}$	0							
	$\hat{k}_{32}, k_{3,3,\iota}$	0							
	$\hat{k}_3$	0	$\widehat{I}_2(k_3) \equiv 0$	$\widehat{I}_2(k_3) \equiv 0$		$\hat{\pi}_2^\times(k_3) \equiv 0$	$\hat{\pi}_2^+(k_3) \equiv 0$		
$\varepsilon = -1$	$\hat{k}_{31}$	0							
	$\hat{k}_{32}, k_{3,3,\iota}$	0							
	$\hat{h}_{1a}$	0			$\widehat{I}_1(h_{1a}) \equiv 0$			$\hat{\pi}_a^+(h_{1a})$	$\hat{\pi}_1^+(h_{1a})$
	$\hat{h}_{1a,1}$	0							
	$\hat{h}_{1b}$	0			$\widehat{I}_1(h_{1b}) \equiv 0$				$\hat{\pi}_1^+(h_{1b}) \equiv 0$
	$\hat{h}_{1b,1}$	0							
	$\hat{h}_{2a}$	0	$\widehat{I}_2(h_{2a}) \equiv 0$	$\widehat{I}_2(h_{2a}) \equiv 0$		$\hat{\pi}_2^\times(h_{2a}) \equiv 0$	$\hat{\pi}_2^+(h_{2a}) \equiv 0$	$\hat{\pi}_a^+(h_{2a})$	
	$\hat{h}_{2a,1}$	0							
	$\hat{h}_{2b}$	0	$\widehat{I}_2(h_{2b}) \equiv 0$	$\widehat{I}_2(h_{2b}) \equiv 0$		$\hat{\pi}_2^\times(h_{2b}) \equiv 0$	$\hat{\pi}_2^+(h_{2b})$		
	$\hat{h}_{2b,1}$	0							$\hat{\pi}_1^+(h_1) \equiv 0$
	$\hat{h}_1$	0			$\widehat{I}_1(h_1) \equiv 0$				
	$\hat{h}_2$	0	$\widehat{I}_2(h_2) \equiv 0$	$\widehat{I}_2(h_2) \equiv 0$		$\hat{\pi}_2^\times(h_2) \equiv 0$	$\hat{\pi}_2^+(h_2) \equiv 0$		
	$\hat{h}_a$	0						$\hat{\pi}_a^+(h_a) \equiv 0$	
	$\hat{h}_b$	0							
	$\hat{h}_3$	0							
	$\hat{h}_6$	0	0	0	0	0	0	0	0

TABLE 2 (b)
$q \equiv -1 \pmod 4$
$G_2(q) =$ Central char. on $2'$-elements $\pmod 2$
$q-1 = 2r'$, $q+1 = 2^d \cdot r$, r and r' odd

	defect:	2	$2d$	$d+1$	$d+2$	$2d+1$	$d+1$	$d+2$	$2d+1$
	$2, 3 \nmid q$	ω_1	ω_2	ω_a	$\omega_{1a}, \omega'_{1a}$	$\omega_{2a}, \omega'_{2a}$	ω_b	$\omega_{1b}, \omega'_{1b}$	$\omega_{2b}, \omega'_{2b}$
	1	1	1	1	1	1	1	1	1
	$\hat{u}_1$								
	$\hat{u}_2$								
	$\hat{u}_3$								
	$\hat{u}_4$								
	$\hat{u}_5$								
	$\hat{u}_6$								
	$\hat{k}_3$	$\hat{\pi}_1(k_3)$					$\hat{\pi}_b(k_3)$	$\hat{\pi}_b^{\#}(k_3)$	
$\varepsilon = 1$	$\hat{k}_{31}$								
	$\hat{k}_{32}, k_{3,3,\iota}$								
	$\hat{k}_3$		$\hat{\pi}_2(k_3)$	$\hat{\pi}_a(k_3)$	$\hat{\pi}_a^{\#}(k_3)$	$\hat{\pi}_2^{*a}(k_3)$			$\hat{\pi}_2^{*b}(k_3)$
$\varepsilon = -1$	$\hat{k}_{31}$								
	$\hat{k}_{32}, k_{3,3,\iota}$								
	$\hat{h}_{1a}$	$\hat{\pi}_1(h_{1a})$		$\hat{\pi}_a(h_{1a})$	$\hat{\pi}_a^{\#}(h_{1a})$				
	$\hat{h}_{1a,1}$								
	$\hat{h}_{1b}$	$\hat{\pi}_1(h_{1b})$					$\hat{\pi}_b(h_{1b})$	$\hat{\pi}_b^{\#}(h_{1b})$	
	$\hat{h}_{1b,1}$								
	$\hat{h}_{2a}$		$\hat{\pi}_2(h_{2a})$	$\hat{\pi}_a(h_{2a})$	$\hat{\pi}_a^{\#}(h_{2a})$	$\hat{\pi}_2^{*a}(h_{2a}) \equiv 0$			$\hat{\pi}_2^{*b}(h_{2a})$
	$\hat{h}_{2a,1}$								
	$\hat{h}_{2b}$		$\hat{\pi}_2(h_{2b})$			$\hat{\pi}_2^{*a}(h_{2b})$	$\hat{\pi}_b(h_{2b})$	$\hat{\pi}_b^{\#}(h_{2b})$	$\hat{\pi}_2^{*b}(h_{2b}) \equiv 0$
	$\hat{h}_{2b,1}$								
	$\hat{h}_1$	$\hat{\pi}_1(h_1)$							
	$\hat{h}_2$		$\hat{\pi}_2(h_2)$			$\hat{\pi}_2^{*a}(h_2)$			$\hat{\pi}_2^{*b}(h_2)$
	$\hat{h}_a$			$\hat{\pi}_a(h_a)$	$\hat{\pi}_a^{\#}(h_a)$				
	$\hat{h}_b$						$\hat{\pi}_b(h_b)$	$\hat{\pi}_b^{\#}(h_b)$	
	$\hat{h}_3$								
	$\hat{h}_6$	0	0	0	0	0	0	0	0

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